

Wilf's Conjecture for numerical semigroups *

Mariam Dhayni [†]

Abstract: Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $c = f + 1 = qm - \rho$ for some $q, \rho \in \mathbb{N}$ with $\rho < m$. Let $\text{Ap}(S, m) = \{w_0 < w_1 < \dots < w_{m-1}\}$ be the Apéry set of S . The aim of this paper is to prove Wilf's Conjecture in some special cases. First, we prove that if $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m - 1$, then S satisfies Wilf's Conjecture. Then, we prove the conjecture in the following cases: $(2 + \frac{1}{q})\nu \geq m$, $m - \nu \leq 5$ and $m = 9$. Finally, the conjecture is proved if $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$ for some $1 < \alpha < m - 1$.

1 Introduction and notations

Let \mathbb{N} denote the set of natural numbers, including 0. A *numerical semigroup* S is an additive submonoid of $(\mathbb{N}, +)$ of finite complement in \mathbb{N} , that is $0 \in S$, if $a, b \in S$ then $a + b \in S$ and $\mathbb{N} \setminus S$ is a finite set. The elements of $\mathbb{N} \setminus S$ are called the *gaps* of S . The largest gap is denoted by $f = f(S) = \max(\mathbb{N} \setminus S)$ and is called the *Frobenius number* of S . The smallest non zero element $m = m(S) = \min(S^*)$ is called the *multiplicity* of S and $n = |\{s \in S, s < f(S)\}|$ is also denoted by $n(S)$. Every numerical semigroup S is finitely generated, i.e. is of the form

$$S = \langle g_1, \dots, g_\nu \rangle = \mathbb{N}g_1 + \dots + \mathbb{N}g_\nu$$

for suitable unique coprime integers g_1, \dots, g_ν . The number of generators of S is denoted by $\nu = \nu(S)$ and is called the *embedding dimension* of S . An integer $x \in \mathbb{N} \setminus S$ is called a *pseudo-Frobenius number* if $x + S^* \subseteq S$. The *type* of the semigroup, denoted by $t(S)$ is the cardinality of set of pseudo-frobenius numbers. The *Apéry set* of S with respect to $a \in S$ is defined as $\text{Ap}(S, a) = \{s \in S; s - a \notin S\}$.

The *Diophantine Frobenius Problem*, also called the *Coin Problem*, is to find the largest number f that cannot be written in the form $\sum_{i=1}^n a_i x_i$; $x_i \in \mathbb{N}$ for given coprime positive integers a_1, \dots, a_n . This problem is related to the theory of numerical semigroups in the following way: let S be the numerical semigroup generated by g_1, \dots, g_ν , then f is simply the largest integer not belonging to S . Hence the problem is to find a formula for f in terms of the set of generators of S . For $\nu = 2$, the formula of $f(\langle g_1, g_2 \rangle)$ is given by Sylvester ([10]), for $\nu \geq 3$ the problem is much harder. It has been proved in ([7]), that in general $f(S)$ is not algebraic in the set of generators of S . In ([11]) 1978 H. S. Wilf proposed a lower bound for the number of generators of S in terms of the Frobenius number as follows: $f(S) + 1 \leq \nu(S)n(S)$.

Although the problem has been considered by several authors (cf. [1], [2], [3], [4], [5], [6], [9], [12]), only special cases have been solved and it remains wide open. In ([3]), D. Dobbs and G. Matthews proved Wilf's Conjecture for $\nu \leq 3$. In ([6]) N. Kaplan proved it for $f + 1 \leq 2m$ and in ([4]) S. Eliahou extended Kaplan's work for $f + 1 \leq 3m$.

*2000 Mathematical Subject Classification: 14H20

Keywords: Wilf's Conjecture, Frobenius number

[†]Université d'Angers, Mathématiques, 49045 Angers cedex 01, France, e-mail:dhayni@math.univ-angers.fr

This work is a generalisation of the case covered by A. Sammartano ([9]), who showed that Wilf's Conjecture holds for $2\nu \geq m$ and $m \leq 8$, based on the idea of counting the elements of S in some intervals of length m . We use different intervals in order to get an equivalent form of Wilf's Conjecture and then to prove it in more cases. We also cover the case where $2\nu \geq m$.

Here are few more details on the contents of this paper. Section 2 is devoted to give some notations that will enable us in the same section to give an equivalent form of Wilf's Conjecture. Section 3 is the heart of the paper. Let $\text{Ap}(S, m) = \{0 = w_0 < w_1 < \dots < w_{m-1}\}$. First, we show that Wilf's conjecture holds for numerical semigroups that satisfy $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m-1$ where $f+1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m-1$. Then we prove Wilf's Conjecture for numerical semigroups with $m - \nu \leq 4$ in order to cover the case where $2\nu \geq m$, prove by Sammartano in ([9]). We also show that a numerical semigroup with $m - \nu = 5$ verify Wilf's Conjecture in order to prove the conjecture for $m = 9$. Finally, we show in this section, using the previous cases, that Wilf's conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$. In section 4 we prove Wilf's Conjecture for numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ and $(\frac{\alpha+3}{3})\nu \geq m$ for some $1 < \alpha < m-1$.

A good reference on numerical semigroups is [8].

2 Preliminaries

Let the notations be as in the introduction. For the sake of clarity we shall use the notations ν, f, n, \dots for $\nu(S), f(S), n(S), \dots$. In this section we will introduce some notations and family of numbers that will enable us to give an equivalent form of Wilf's conjecture.

Definition. Let S be a numerical semigroup and let $c = C(S) = f + 1$ be the conductor of S . Denote by

$$q = \lceil \frac{c}{m} \rceil,$$

where $\lceil x \rceil$ denote the smallest integer greater than or equal to x . Thus, $qm \geq c$ and $c = qm - \rho$ with $0 \leq \rho < m$. Given a non negative integer k , we define the k th interval of length m ,

$$I_k = [km - \rho, (k+1)m - \rho[= \{km - \rho, km - \rho + 1, \dots, (k+1)m - \rho - 1\}.$$

We denote by

$$n_k = |\{s \in S \cap I_k\}|.$$

For $j \in \{1, \dots, m-1\}$, we define η_j to be the number of intervals I_k with $n_k = j$.

$$\eta_j = |\{k \in \mathbb{N}; |I_k \cap S| = j\}|.$$

Proposition 2.1 Under the previous notations, we have:

- i) $1 \leq n_k \leq m-1$ for all $k = 0, \dots, q-1$.
- ii) $n_k = m$ for all $k \geq q$.
- iii) $n = n(S) = \sum_{k=0}^{q-1} n_k$.
- iv) $\sum_{j=1}^{m-1} \eta_j = q$.
- v) $\sum_{j=1}^{m-1} j\eta_j = \sum_{k=0}^{q-1} n_k = n$.

Proof. *i), ii), iii)* are obvious. Now using *i), ii)* we will prove *iv)* and *v)*.

$$iv) \sum_{j=1}^{m-1} \eta_j = \sum_{j=1}^{m-1} |\{k \in \mathbb{N}; |I_k \cap S| = j\}| = \sum_{j=1}^{m-1} |\{k \in \mathbb{N}; n_k = j; k = 0, \dots, q-1\}| = q.$$

$$v) \sum_{j=1}^{m-1} j\eta_j = \sum_{j=1}^{m-1} j|\{k \in \mathbb{N}; |I_k \cap S| = j\}| = \sum_{j=1}^{m-1} j|\{k \in \mathbb{N}; n_k = j; k = 0, \dots, q-1\}| = \sum_{k=0}^{q-1} n_k = n. \quad \square$$

Remark: We shall use the notation $\lfloor x \rfloor$ for the largest integer smaller than or equal to x .

Next we will express η_j in terms of the Apéry set.

Proposition 2.2 Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}\}$. Under the previous notations, we have for all $j \in \{1, \dots, m-1\}$

$$\eta_j = \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor.$$

Proof. Fix $0 \leq k \leq q-1$, and let $j \in \{1, \dots, m-1\}$. We will show that the interval I_k contains exactly j elements of S if and only if $w_{j-1} < (k+1)m - \rho \leq w_j$. Recall to this end that for all $s \in S$, there exist $0 \leq i \leq m-1, a \in \mathbb{N}$ such that $s = w_i + am$.

Suppose that I_k contains j elements. Suppose, by contradiction, that $w_{j-1} \geq (k+1)m - \rho$. We have $w_{m-1} > \dots > w_{j-1} \geq (k+1)m - \rho$, thus $w_{m-1}, \dots, w_{j-1} \in \cup_{t=k+1}^q I_t$. Hence, I_k contains at most $j-1$ elements of S (namely $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-2} + k_{j-2}m$ for some $k_1, \dots, k_{j-2} \in \{0, \dots, k-1\}$). This contradicts the fact that I_k contains exactly j elements of S .

If $w_j < (k+1)m - \rho$, then $w_0 < \dots < w_j < (k+1)m - \rho$, thus $w_0, \dots, w_j \in \cup_{t=0}^k I_t$. Hence, I_k contains at least $j+1$ elements of S which are: $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_j + k_jm$ for some $k_1, \dots, k_j \in \{0, \dots, k-1\}$, which contradicts the fact that I_k contains exactly j elements of S .

Conversely, $w_{j-1} < (k+1)m - \rho$ implies that $w_0 < \dots < w_{j-1} < (k+1)m - \rho$, so $w_0, \dots, w_{j-1} \in \cup_{t=0}^k I_t$. Thus I_k contains at least j elements which are: $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-1} + k_{j-1}m$ for some $k_1, \dots, k_{j-1} \in \{0, \dots, k-1\}$.

On the other hand $w_j \geq (k+1)m - \rho$ implies that $w_{m-1} > \dots > w_j \geq (k+1)m - \rho$, so $w_{m-1}, \dots, w_j \in \cup_{t=k+1}^q I_t$. Thus I_k contains at most j elements which are: $w_0 + km = km, w_1 + k_1m, w_2 + k_2m, \dots, w_{j-1} + k_{j-1}m$ for some $k_1, \dots, k_{j-1} \in \{0, \dots, k-1\}$. Hence, if $w_{j-1} < (k+1)m - \rho \leq w_j$, then I_k contains exactly j elements of S and this proves our assertion.

Consequently,

$$\begin{aligned} \eta_j &= |\{k \in \mathbb{N} \text{ such that } |I_k \cap S| = j\}| \\ &= |\{k \in \mathbb{N} \text{ such that } w_{j-1} < (k+1)m - \rho \leq w_j\}| \\ &= |\{k \in \mathbb{N} \text{ such that } \frac{w_{j-1} + \rho}{m} < (k+1) \leq \frac{w_j + \rho}{m}\}| \\ &= |\{k \in \mathbb{N} \text{ such that } \frac{w_{j-1} + \rho}{m} - 1 < k \leq \frac{w_j + \rho}{m} - 1\}| \\ &= |\{k \in \mathbb{N} \text{ such that } \lfloor \frac{w_{j-1} + \rho}{m} \rfloor \leq k \leq \lfloor \frac{w_j + \rho}{m} \rfloor - 1\}| \\ &= \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor. \quad \square \end{aligned}$$

Proposition 2.3 gives an equivalent form of Wilf's Conjecture using Propositions 2.1 and 2.2.

Proposition 2.3 Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$ and $0 \leq \rho \leq m - 1$. Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Then S satisfies Wilf's Conjecture *if and only if*

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Proof. By Proposition 2.1, we have

$$f + 1 \leq n\nu \Leftrightarrow qm - \rho \leq \nu \sum_{k=0}^{q-1} n_k \Leftrightarrow \sum_{k=0}^{q-1} m - \rho \leq \sum_{k=0}^{q-1} n_k \nu \Leftrightarrow \sum_{k=0}^{q-1} (n_k \nu - m) + \rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1} \eta_j(j\nu - m) + \rho \geq 0.$$

Using Proposition 2.2, we have

$$\sum_{j=1}^{m-1} \eta_j(j\nu - m) + \rho \geq 0 \Leftrightarrow \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

Thus the proof is complete. \square

Remark 2.4 Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < \dots < w_{m-1}\}$. The following technical results will be used through the paper:

1. $\lfloor \frac{w_0 + \rho}{m} \rfloor = 0$ ($w_0 = 0$ and $0 \leq \rho < m$).
2. For all $1 \leq i \leq m - 1$, $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 1$ ($w_i > m$).
3. For all $1 \leq i \leq m - 1$, either $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor$ or $\lfloor \frac{w_i + \rho}{m} \rfloor = \lfloor \frac{w_i}{m} \rfloor + 1$. In the second case $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 2$ and $\rho \geq 1$.
4. If $w_i < w_j$ for some $0 \leq i < j \leq m - 1$, then $\lfloor \frac{w_i + \rho}{m} \rfloor \leq \lfloor \frac{w_j + \rho}{m} \rfloor$.
5. $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor = \lfloor \frac{qm - \rho + m - 1 + \rho}{m} \rfloor = q$.

3 Main Results

In this section, we show that Wilf's Conjecture holds for numerical semigroups in the following cases:

1. $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$ for some $1 < \alpha < m - 1$.
2. $m - \nu \leq 5$. (Note that the case $m - \nu \leq 4$ results from the fact that Wilf's Conjecture holds for $2\nu \geq m$ ([9]), however we shall give the proof for $m - \nu \leq 3$ in order to cover this result through our techniques).

We then deduce the conjecture for $m = 9$ and for $(2 + \frac{1}{q})\nu \geq m$.

The following technical Lemma will be used through the paper:

Lemma 3.1 Let $\text{Ap}(S, m) = \{w_0 = 0 < w_1 < \dots < w_{m-1}\}$. Suppose that $w_i \geq w_j + w_k$, then $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor - 1$. If furthermore, $\lfloor \frac{w_i + \rho}{m} \rfloor - \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_k + \rho}{m} \rfloor = -1$, then $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1$, $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1$ and $\rho \geq 1$. In particular, $\lfloor \frac{w_i + \rho}{m} \rfloor \geq 2$, $\lfloor \frac{w_k + \rho}{m} \rfloor \geq 2$ and $\rho \geq 1$.

Proof. $w_i \geq w_j + w_k$ implies that $w_i + \rho \geq w_j + w_k + \rho$, hence $\frac{w_i + \rho}{m} \geq \frac{w_j + w_k + \rho}{m}$. Consequently, $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + w_k + \rho}{m} \rfloor$. Therefore, $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k}{m} \rfloor$. Hence, by Remark 2.4 (3), $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor - 1$. Suppose that $w_i \geq w_j + w_k$ and that $\lfloor \frac{w_i + \rho}{m} \rfloor - \lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_k + \rho}{m} \rfloor = -1$. Suppose by the way of contradiction that either $\lfloor \frac{w_j + \rho}{m} \rfloor \neq \lfloor \frac{w_j}{m} \rfloor + 1$ or $\lfloor \frac{w_k + \rho}{m} \rfloor \neq \lfloor \frac{w_k}{m} \rfloor + 1$ or $\rho < 1$. Then, by Remark 2.4 (3) and the fact that $\rho \geq 0$, we have either $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor$ or $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor$ or $\rho = 0$. Since $w_i \geq w_j + w_k$, then $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + w_k + \rho}{m} \rfloor$. In this case $\lfloor \frac{w_i + \rho}{m} \rfloor \geq \lfloor \frac{w_j + \rho}{m} \rfloor + \lfloor \frac{w_k + \rho}{m} \rfloor$, which is impossible. Hence, $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1$, $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1$ and $\rho \geq 1$. Therefore, by Remark 2.4 (2), $\lfloor \frac{w_j + \rho}{m} \rfloor = \lfloor \frac{w_j}{m} \rfloor + 1 \geq 2$, $\lfloor \frac{w_k + \rho}{m} \rfloor = \lfloor \frac{w_k}{m} \rfloor + 1 \geq 2$ and $\rho \geq 1$. \square

Next we will show that Wilf's Conjecture holds for numerical semigroups with $w_{m-1} \geq w_1 + w_\alpha$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$.

Theorem 3.2 Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m - 1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3. We have,

$$\begin{aligned}
\sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=1}^{\alpha} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor(j\nu - m) \\
&= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=0}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor((j+1)\nu - m) \\
&= \lfloor \frac{w_\alpha + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor(\nu - m) - \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor\nu \\
(3.1) \quad &= \lfloor \frac{w_\alpha + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor\nu - \sum_{j=2}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor\nu \\
&\geq \lfloor \frac{w_\alpha + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor\nu - \sum_{j=2}^{\alpha-1} \lfloor \frac{w_\alpha + \rho}{m} \rfloor\nu \\
&= \lfloor \frac{w_\alpha + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_1 + \rho}{m} \rfloor\nu - \lfloor \frac{w_\alpha + \rho}{m} \rfloor(\alpha - 2)\nu \\
&= -\lfloor \frac{w_1 + \rho}{m} \rfloor\nu + \lfloor \frac{w_\alpha + \rho}{m} \rfloor(2\nu - m).
\end{aligned}$$

$$\begin{aligned}
\sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) &\geq \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)((\alpha+1)\nu - m) \\
&= ((\alpha+1)\nu - m) \sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\
(3.2) \quad &= ((\alpha+1)\nu - m) (\sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor) \\
&= ((\alpha+1)\nu - m) (\sum_{j=\alpha+1}^{m-1} \lfloor \frac{w_j + \rho}{m} \rfloor - \sum_{j=\alpha}^{m-2} \lfloor \frac{w_j + \rho}{m} \rfloor) \\
&= (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha+1)\nu - m).
\end{aligned}$$

Since $w_{m-1} \geq w_1 + w_\alpha$, by Lemma 3.1, it follows that $\lfloor \frac{w_{m-1}+\rho}{m} \rfloor \geq \lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor - 1$. Let $x = \lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor$. Then, $\lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor = q - x$ and $x \geq -1$. Now using (3.1) and (3.2), we have

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j+\rho}{m} \rfloor - \lfloor \frac{w_{j-1}+\rho}{m} \rfloor)(j\nu - m) + \rho &\geq -\lfloor \frac{w_1+\rho}{m} \rfloor \nu + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor (-\nu + ((\alpha+1)\nu - m) - ((\alpha+1)\nu - m)) + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
&= \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha\nu - m) + \lfloor \frac{w_\alpha+\rho}{m} \rfloor (2\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
&= (\lfloor \frac{w_1+\rho}{m} \rfloor + \lfloor \frac{w_\alpha+\rho}{m} \rfloor)(2\nu - m) + \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha - 2)\nu \\
&\quad + (\lfloor \frac{w_{m-1}+\rho}{m} \rfloor - \lfloor \frac{w_\alpha+\rho}{m} \rfloor - \lfloor \frac{w_1+\rho}{m} \rfloor)((\alpha+1)\nu - m) + \rho \\
&= (q - x)(2\nu - m) + \lfloor \frac{w_1+\rho}{m} \rfloor (\alpha - 2)\nu + x((\alpha+1)\nu - m) + \rho \\
&\geq (q - x)(2\nu - m) + (\alpha - 2)\nu + x((\alpha+1)\nu - m) + \rho \\
&= \nu(2q - 2x + \alpha - 2 + x\alpha + x) - qm + \rho \\
&= \nu(2q + (\alpha - 2)(x + 1) + x) - qm + \rho \\
&\geq \nu(2q + \alpha - 3) - qm + \rho \quad (x \geq -1) \\
&= q(\nu(2 + \frac{\alpha - 3}{q}) - m) + \rho \\
&\geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture. \square

Example 3.3 Consider the following numerical semigroup $S = \langle 19, 21, 23, 25, 27, 28 \rangle$. Note that $3\nu < m$. We have $w_1 = 21$, $w_{14} = 56$ and $w_{m-1} = 83$ i.e. $w_{m-1} \geq w_1 + w_{14}$. In addition, $(2 + \frac{\alpha-3}{q})\nu = (2 + \frac{14-3}{4})6 \geq 19 = m$. Thus the conditions of Theorem 3.2 are valid, so S satisfies Wilf's Conjecture.

In the following we shall deduce some cases where Wilf's Conjecture holds. We start with the following technical Lemma.

Lemma 3.4 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $m - \nu > \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ for some $\alpha \in \mathbb{N}^*$, then $w_{m-1} \geq w_1 + w_\alpha$.

Proof. Recall that an element x of the Apéry set of S belongs to $\min(\text{Ap}(S, m))$ if and only if $x \neq w_i + w_j$ for all $w_i, w_j \in \text{Ap}(S, m) \setminus \{0\}$, in particular $m - \nu = |\text{Ap}(S, m) \setminus \min(\text{Ap}(S, m))|$. Suppose by the way of contradiction

that $w_{m-1} < w_1 + w_\alpha$, and let $w \in \text{Ap}(S, m) \setminus \min(\text{Ap}(S, m))$. Then $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S, m) \setminus \{0\}$. Hence, $w \leq w_{m-1} < w_1 + w_\alpha$. Thus the only possible values for w are $\{w_i + w_j; 1 \leq i \leq j \leq \alpha - 1\}$. Therefore, $m - \nu \leq \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$, which is impossible. Hence, $w_{m-1} \geq w_1 + w_\alpha$. \square

Next we will deduce Wilf's Conjecture for numerical Semigroups with $m - \nu > \frac{\alpha(\alpha-1)}{2}$ and $(2 + \frac{\alpha-3}{q})\nu \geq m$. It will be used later to show that the conjecture holds for those with $(2 + \frac{1}{q})\nu \geq m$, and in order also to cover the result in [9] saying that the conjecture is true for $2\nu \geq m$.

Corollary 3.5 Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. Suppose that $m - \nu > \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$ for some $1 < \alpha < m - 1$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. It follows from Lemma 3.4 that if $m - \nu > \frac{\alpha(\alpha-1)}{2}$, then $w_{m-1} \geq w_1 + w_\alpha$. Now use Theorem 3.2. \square

As a direct consequence of Theorem 3.2, we get the following Corollary.

Corollary 3.6 Let S be a numerical semigroup with a given multiplicity m and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $w_{m-1} \geq w_1 + w_\alpha$ for some $1 < \alpha < m - 1$ and $m \leq 8 + 4(\frac{\alpha-3}{q})$ then S satisfies Wilf's Conjecture.

Proof. By Theorem 3.2, we may assume that $(2 + \frac{\alpha-3}{q})\nu < m$. Therefore, $\nu < \frac{qm}{2q+\alpha-3} \leq \frac{8q+\alpha-12}{2q+\alpha-3}$. Hence $\nu < 4$, consequently S satisfies Wilf's Conjecture ([3]). \square

In the following Lemma, we will show that Wilf's Conjecture holds for numerical semigroups with $m - \nu \leq 3$. This will enable us later to prove the conjecture for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$ and cover the result in [9] saying that the conjecture is true for $2\nu \geq m$.

Lemma 3.7 Let S be a numerical Semigroup with multiplicity m and embedding dimension ν . If $m - \nu \leq 3$, then S satisfies Wilf's Conjecture.

Proof. We may assume that $\nu \geq 4$ ($\nu \leq 3$ is solved [3]). We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3.

i) If $m - \nu = 1$, then we may assume that $m = \nu + 1 \geq 5$. By taking $\alpha = 1$ in (3.2), we get

$$\begin{aligned} \sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) &\geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m). \text{ Hence,} \\ \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &= (\lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_0 + \rho}{m} \rfloor)(\nu - m) \\ &\quad + \sum_{j=2}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\ &\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(\nu - m) \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \\ &= \lfloor \frac{w_1 + \rho}{m} \rfloor(\nu - m + (2\nu - m) - (2\nu - m)) \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \end{aligned}$$

$$\begin{aligned}
&= \lfloor \frac{w_1 + \rho}{m} \rfloor (3\nu - 2m) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) + \rho \\
&= \lfloor \frac{w_1 + \rho}{m} \rfloor (m - 3) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(m - 2) + \rho.
\end{aligned}$$

Since $m - \nu = 1 > 0 = \frac{1(0)}{2}$, then by Lemma 3.4, it follows that $w_{m-1} \geq w_1 + w_1$. Consequently, by Lemma 3.1, we have $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_1 + \rho}{m} \rfloor - 1$.

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor = -1$. Then by Lemma 3.1, we have $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2$. Since $m \geq 5$, then
$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 2(m - 3) - (m - 2) + \rho \geq 0.$$
- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor \geq 0$. Since $m \geq 5$, then
$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq (m - 3) + \rho \geq 0.$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m - \nu = 1$.

ii) If $m - \nu \in \{2, 3\}$. By taking $\alpha = 2$ in (3.2), we get

$$\begin{aligned}
\sum_{j=3}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) &\geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m). \quad \text{Hence,} \\
\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &= (\lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_0 + \rho}{m} \rfloor)(\nu - m) \\
&+ (\lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) \\
&+ \sum_{j=3}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
&\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_2 + \rho}{m} \rfloor(2\nu - m) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m) + \rho \\
(3.3) \quad &= \lfloor \frac{w_1 + \rho}{m} \rfloor(-\nu + (3\nu - m) - (3\nu - m)) \\
&+ \lfloor \frac{w_2 + \rho}{m} \rfloor(2\nu - m) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m) + \rho \\
&= \lfloor \frac{w_1 + \rho}{m} \rfloor(2\nu - m) + \lfloor \frac{w_2 + \rho}{m} \rfloor(2\nu - m) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m) \\
&+ \rho.
\end{aligned}$$

Since $m - \nu \in \{2, 3\} > 1$, by Lemma 3.4, we have $w_{m-1} \geq w_1 + w_2$. It follows from Lemma 3.1 that $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_1 + \rho}{m} \rfloor + \lfloor \frac{w_2 + \rho}{m} \rfloor - 1$.

- If $m - \nu = 2$. then we may assume that $m = \nu + 2 \geq 6$. Now (3.3) gives,

$$\begin{aligned} \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(m-4) + \lfloor \frac{w_2 + \rho}{m} \rfloor(m-4) \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(2m-6) + \rho. \end{aligned}$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor = -1$. Then by Lemma 3.1 we have, $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2$ and $\lfloor \frac{w_2 + \rho}{m} \rfloor \geq 2$. Since $m \geq 6$, then

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 2(m-4) + 2(m-4) - (2m-6) + \rho \geq 0.$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor \geq 0$. Since $m \geq 6$, then

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq (m-4) + (m-4) + \rho \geq 0.$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m - \nu = 2$.

- If $m - \nu = 3$, then we may assume that $m = \nu + 3 \geq 7$. Now (3.3) gives,

$$\begin{aligned} \sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(m-6) + \lfloor \frac{w_2 + \rho}{m} \rfloor(m-6) \\ &\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(2m-9) + \rho. \end{aligned}$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor = -1$. Then by Lemma 3.1 we have, $\lfloor \frac{w_1 + \rho}{m} \rfloor \geq 2$, $\lfloor \frac{w_2 + \rho}{m} \rfloor \geq 2$ and $\rho \geq 1$. Since $m \geq 7$, then

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 2(m-6) + 2(m-6) - (2m-9) + 1 \geq 0.$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor \geq 0$. Since $m \geq 7$, then

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq (m-6) + (m-6) + \rho \geq 0.$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture if $m - \nu = 3$.

Thus Wilf's Conjecture holds if $m - \nu \leq 3$. \square

The next Corollary covers the result of Sammartano for numerical semigroups with $2\nu \geq m$ ([9]) using Corollary 3.5 and Lemma 3.7.

Corollary 3.8 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $2\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. If $m - \nu > 3$ and $2\nu \geq m$, then by Corollary 3.5 Wilf's Conjecture holds. If $m - \nu \leq 3$, by Lemma 3.7, S satisfies Wilf's Conjecture. \square

In the following Corollary we will deduce Wilf's Conjecture for numerical semigroups with $m - \nu = 4$. This will enable us later to prove the conjecture for those with $(2 + \frac{1}{q})\nu \geq m$.

Corollary 3.9 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $m - \nu = 4$, then S satisfies Wilf's Conjecture.

Proof. Since Wilf's conjecture holds for $\nu \leq 3$ ([3]), then we may assume that $\nu \geq 4$. Hence, $\nu \geq m - \nu$. Consequently, $2\nu \geq m$. Hence, S satisfies Wilf's Conjecture. \square

The following technical Lemma will be used through the paper.

Lemma 3.10 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. If $m - \nu \geq \binom{\alpha}{2} - 1 = \frac{\alpha(\alpha-1)}{2} - 1$ for some $3 \leq \alpha \leq m - 2$, then $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$.

Proof. Suppose by the way of contradiction that $w_{m-1} < w_1 + w_\alpha$ and $w_{m-1} < w_{\alpha-2} + w_{\alpha-1}$. Let $w \in \text{Ap}(S, m) \setminus \min(\text{Ap}(S, m))$, then $w \leq w_{m-1}$ and $w = w_i + w_j$ for some $w_i, w_j \in \text{Ap}(S, m) \setminus \{0\}$. In this case, the only possible values of w are $\{w_i + w_j; 1 \leq i \leq j \leq \alpha - 1\} \setminus \{w_{\alpha-2} + w_{\alpha-1}, w_{\alpha-1} + w_{\alpha-1}\}$. Consequently, $m - \nu = |\text{Ap}(S, m) \setminus \min(\text{Ap}(S, m))| \leq \frac{\alpha(\alpha-1)}{2} - 2$. But $\frac{\alpha(\alpha-1)}{2} - 2 < \frac{\alpha(\alpha-1)}{2} - 1$, which contradicts the hypothesis. Hence, $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. \square

In the next theorem, we will show that Wilf's Conjecture holds for numerical semigroups with $m - \nu = 5$.

Theorem 3.11 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . If $m - \nu = 5$, then S satisfies Wilf's Conjecture.

Proof. Let $m - \nu = 5$. Since Wilf's Conjecture holds for $2\nu \geq m$, then we may assume that $2\nu < m$. This implies that $\nu < 5$. Since the case $\nu \leq 3$ is known ([3]), then we shall assume that $\nu = 4$. This also implies that $m = 9$.

Since $m - \nu = 5 = \frac{4(3)}{2} - 1$, by Lemma 3.10, it follows that $w_8 \geq w_2 + w_3$ or $w_8 \geq w_1 + w_4$.

i) If $w_8 \geq w_2 + w_3$. By taking $\alpha = 3$ in (3.2) ($m = 9, \nu = 4$), we get

$$\sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) \geq (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(16 - 9) = (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7). \quad (7)$$

Hence,

$$\begin{aligned} \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho &= (\lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_0 + \rho}{9} \rfloor)(-5) \\ &\quad + (\lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(-1) \\ &\quad + (\lfloor \frac{w_3 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor)(3) \\ &\quad + \sum_{j=4}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \\ (3.4) \quad &\geq \lfloor \frac{w_1 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(3) \\ &\quad + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \\ &\geq \left(\lfloor \frac{w_2 + \rho}{9} \rfloor \left(\left(\frac{-3}{4} \right) 4 \right) + \lfloor \frac{w_3 + \rho}{9} \rfloor \left(\left(\frac{-1}{4} \right) 4 \right) \right) \\ &\quad + \lfloor \frac{w_2 + \rho}{9} \rfloor(-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor(3) \\ &\quad + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor)(7) + \rho \end{aligned}$$

$$\begin{aligned}
&= \lfloor \frac{w_2 + \rho}{9} \rfloor (-7) + \lfloor \frac{w_3 + \rho}{9} \rfloor (2) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor) (7) + \rho \\
&= \lfloor \frac{w_3 + \rho}{9} \rfloor (2) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor) (7) + \rho.
\end{aligned}$$

Since $w_8 \geq w_2 + w_3$, by Lemma 3.1, it follows that $\lfloor \frac{w_8 + \rho}{9} \rfloor \geq \lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor - 1$.

- If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor \geq 0$, then (3.4) gives $\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor) (4j - 9) + \rho \geq 0$.
- If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor = -1$. By Lemma 3.1, we have $\rho \geq 1$.
Since for $q \leq 3$ Wilf's Conjecture is solved ([4], [6]), then may assume that $q \geq 4$. Since $\lfloor \frac{w_2 + \rho}{9} \rfloor \leq \lfloor \frac{w_3 + \rho}{9} \rfloor$ and $\lfloor \frac{w_2 + \rho}{9} \rfloor + \lfloor \frac{w_3 + \rho}{9} \rfloor = \lfloor \frac{w_8 + \rho}{9} \rfloor + 1 = q + 1$, in this case it follows that $\lfloor \frac{w_3 + \rho}{9} \rfloor \geq 3$.
Now (3.4) gives, $\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor) (4j - m) + \rho \geq 3(2) - 7 + 1 \geq 0$.

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

ii) If $w_8 \geq w_1 + w_4$. We may assume that $w_8 < w_2 + w_3$, since otherwise we are back to case 1. Hence, the possible values of $w \in \text{Ap}(S, 9) \setminus \min(\text{Ap}(S, 9))$ are $\{w_1 + w_j; 1 \leq j \leq 7\} \cup \{w_2 + w_2\}$.

- Recall that an element x of the Apéry set of S belongs to $\max(\text{Ap}(S, m))$ if and only if $w_i \neq x + w_j$ for all $w_i, w_j \in \text{Ap}(S, m) \setminus \{0\}$. If $\text{Ap}(S, 9) \setminus \min(\text{Ap}(S, 9)) \subseteq \{w_1 + w_j; 1 \leq j \leq 7\}$, then there exists at least five elements in $\text{Ap}(S, 9)$ that are not maximal, hence $t(S) = |\{\max(\text{Ap}(S, 9)) - 9\}| \leq 3 = \nu - 1$. Consequently, S satisfies Wilf's Conjecture ([3] Proposition 2.3).
- If $w_2 + w_2 \in \text{Ap}(S, 9) \setminus \min(\text{Ap}(S, 9))$, then $w_8 \geq w_2 + w_2$. By Lemma 3.1 we have $\lfloor \frac{w_8 + \rho}{9} \rfloor \geq 2\lfloor \frac{w_2 + \rho}{9} \rfloor - 1$. In particular,

$$(3.5) \quad \lfloor \frac{w_2 + \rho}{9} \rfloor \leq \frac{q+1}{2}.$$

By taking $\alpha = 4$ in (3.2) ($m = 9, \nu = 4$), we get

$$\sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor) (4j - 9) \geq (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor) (11). \quad \text{Now using (3.5), we get}$$

$$\begin{aligned}
(3.6) \quad \sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor) (4j - 9) + \rho &= (\lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_0 + \rho}{9} \rfloor) (-5) \\
&+ (\lfloor \frac{w_2 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor) (-1) \\
&+ (\lfloor \frac{w_3 + \rho}{9} \rfloor - \lfloor \frac{w_2 + \rho}{9} \rfloor) (3) \\
&+ (\lfloor \frac{w_4 + \rho}{9} \rfloor - \lfloor \frac{w_3 + \rho}{9} \rfloor) (7) \\
&+ \sum_{j=5}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor) (4j - 9) + \rho
\end{aligned}$$

$$\begin{aligned}
&\geq \lfloor \frac{w_1 + \rho}{9} \rfloor (-4) + \lfloor \frac{w_2 + \rho}{9} \rfloor (-4) + \lfloor \frac{w_3 + \rho}{9} \rfloor (-4) \\
&+ \lfloor \frac{w_4 + \rho}{9} \rfloor (7) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
&\geq \lfloor \frac{w_1 + \rho}{9} \rfloor (-4) + (\frac{q+1}{2})(-4) + \lfloor \frac{w_4 + \rho}{9} \rfloor (-4) \\
&+ \lfloor \frac{w_4 + \rho}{9} \rfloor (7) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
&= \lfloor \frac{w_1 + \rho}{9} \rfloor (-4) - 2(q+1) + \lfloor \frac{w_4 + \rho}{9} \rfloor (3) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
&= \lfloor \frac{w_1 + \rho}{9} \rfloor (-4 + 11 - 11) - 2(q+1) \\
&+ \lfloor \frac{w_4 + \rho}{9} \rfloor (3) + (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho \\
&= \lfloor \frac{w_1 + \rho}{9} \rfloor (7) - 2(q+1) + \lfloor \frac{w_4 + \rho}{9} \rfloor (3) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor)(11) + \rho \\
&= (\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor)(3) + \lfloor \frac{w_1 + \rho}{9} \rfloor (4) - 2(q+1) \\
&+ (\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor)(11) + \rho.
\end{aligned}$$

We have $w_8 \geq w_1 + w_4$, then by Lemma 3.1 $\lfloor \frac{w_8 + \rho}{9} \rfloor \geq \lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor - 1$.

– If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor \geq 0$. Let $x = \lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor$. Hence, $x \geq 0$ and $\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor = q - x$. Then (3.6) gives,

$$\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \geq (q - x)(3) + 4 - 2(q + 1) + 11x + \rho = q + 8x + 2 + \rho \geq 0.$$

– If $\lfloor \frac{w_8 + \rho}{9} \rfloor - \lfloor \frac{w_1 + \rho}{9} \rfloor - \lfloor \frac{w_4 + \rho}{9} \rfloor = -1$. Then $\lfloor \frac{w_1 + \rho}{9} \rfloor + \lfloor \frac{w_4 + \rho}{9} \rfloor = q + 1$. By Lemma 3.1, we have $\lfloor \frac{w_1 + \rho}{9} \rfloor \geq 2$ and $\rho \geq 1$. Since $q \geq 1$, then (3.6) gives,

$$\sum_{j=1}^8 (\lfloor \frac{w_j + \rho}{9} \rfloor - \lfloor \frac{w_{j-1} + \rho}{9} \rfloor)(4j - 9) + \rho \geq (q + 1)(3) + 8 - 2(q + 1) - 11 + 1 = q - 1 \geq 0.$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

Thus, Wilf's Conjecture holds if $m - \nu = 5$. \square

In the next corollary, we will deduce the conjecture for $m = 9$.

Corollary 3.12 If S is a numerical Semigroup with multiplicity $m = 9$, then S satisfies Wilf's Conjecture.

Proof. By Lemma 3.7, Corollary 3.9 and Theorem 3.11, we may assume that $m - \nu > 5$, hence $\nu < m - 5 = 4$. By ([3]) S satisfies Wilf's Conjecture. \square

The following Lemma will enable us later to show that Wilf's Conjecture holds for numerical semigroups with $(2 + \frac{1}{q})\nu \geq m$.

Lemma 3.13 Let S be a numerical Semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. Since $m - \nu = 6 \geq \frac{4(3)}{2} - 1$, by Lemma 3.10, it follows that $w_{m-1} \geq w_1 + w_4$ or $w_{m-1} \geq w_2 + w_3$.

- i) If $w_{m-1} \geq w_1 + w_4$. By hypothesis $(2 + \frac{1}{q})\nu \geq m$ and Theorem 3.2 Wilf's Conjecture holds in this case.
- ii) If $w_{m-1} \geq w_2 + w_3$. We may assume that $w_{m-1} < w_1 + w_4$, since otherwise we are back to case 1. Hence, $\text{Ap}(S, m) \setminus \min(\text{Ap}(S, m)) = \{w_1 + w_1, w_1 + w_2, w_1 + w_3, w_2 + w_2, w_2 + w_3, w_3 + w_3\}$.

By taking $\alpha = 3$ in (3.2), we get $\sum_{j=4}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m)$.

Hence,

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &= (\lfloor \frac{w_1 + \rho}{m} \rfloor - \lfloor \frac{w_0 + \rho}{m} \rfloor)(\nu - m) \\
&\quad + (\lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_1 + \rho}{m} \rfloor)(2\nu - m) \\
&\quad + (\lfloor \frac{w_3 + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor)(3\nu - m) \\
&\quad + \sum_{j=4}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \\
&\geq \lfloor \frac{w_1 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_2 + \rho}{m} \rfloor(-\nu) \\
&\quad + \lfloor \frac{w_3 + \rho}{m} \rfloor(3\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
&\geq (\lfloor \frac{w_2 + \rho}{m} \rfloor(\frac{-\nu}{2}) + \lfloor \frac{w_3 + \rho}{m} \rfloor(\frac{-\nu}{2})) \\
&\quad + \lfloor \frac{w_2 + \rho}{m} \rfloor(-\nu) + \lfloor \frac{w_3 + \rho}{m} \rfloor(3\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
&= \lfloor \frac{w_2 + \rho}{m} \rfloor(\frac{-3\nu}{2}) + \lfloor \frac{w_3 + \rho}{m} \rfloor(\frac{5\nu}{2} - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
&= \lfloor \frac{w_2 + \rho}{m} \rfloor(\frac{-3\nu}{2} + (4\nu - m) - (4\nu - m)) \\
&\quad + \lfloor \frac{w_3 + \rho}{m} \rfloor(\frac{5\nu}{2} - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) + \rho \\
&= \lfloor \frac{w_2 + \rho}{m} \rfloor(\frac{5\nu}{2} - m) + \lfloor \frac{w_3 + \rho}{m} \rfloor(\frac{5\nu}{2} - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(4\nu - m) \\
&\quad + \rho
\end{aligned}
\tag{3.7}$$

$$\begin{aligned}
&= \lfloor \frac{w_2 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) + \lfloor \frac{w_3 + \rho}{m} \rfloor (\frac{3\nu}{2} - 6) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(3\nu - 6) \\
&+ \rho.
\end{aligned}$$

We have $w_{m-1} \geq w_2 + w_3$, by Lemma 3.1, it follows that $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor - 1$.

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor \geq 0$, using $\nu \geq 4$ in (3.7) ($\nu \leq 3$ is solved [3]), we get

$$\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho \geq 0.$$

- If $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor = -1$. Then,

$$(3.8) \quad \lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor = q + 1.$$

We have $w_3 + w_3 \in \text{Ap}(S, m) \setminus \min(\text{Ap}(S, m))$, then $w_{m-1} \geq w_3 + w_3$. By Lemma 3.1, we have $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq 2\lfloor \frac{w_3 + \rho}{m} \rfloor - 1$. In particular,

$$(3.9) \quad \lfloor \frac{w_3 + \rho}{m} \rfloor \leq \frac{q+1}{2}.$$

Since Wilf's Conjecture holds for $q \leq 3$ ([4], [6]), so we may assume that $q \geq 4$. Since $\lfloor \frac{w_2 + \rho}{m} \rfloor \leq \lfloor \frac{w_3 + \rho}{m} \rfloor$, by (3.8) and (3.9), it follows that $\lfloor \frac{w_2 + \rho}{m} \rfloor = \lfloor \frac{w_3 + \rho}{m} \rfloor = \frac{q+1}{2}$, in particular q is odd, so we have to assume that $q \geq 5$. Now using (3.8), $q \geq 5$ and the hypothesis $(2 + \frac{1}{q})\nu \geq m = \nu + 6$ in (3.7), we get

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &\geq (\lfloor \frac{w_2 + \rho}{m} \rfloor + \lfloor \frac{w_3 + \rho}{m} \rfloor)(\frac{3\nu}{2} - 6) \\
&+ (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_2 + \rho}{m} \rfloor - \lfloor \frac{w_3 + \rho}{m} \rfloor)(3\nu - 6) + \rho \\
&= (q+1)(\frac{3\nu}{2} - 6) - (3\nu - 6) + \rho \\
&= \nu(\frac{3q}{2} + \frac{3}{2} - 3) - 6q + \rho \\
&\geq \nu(\frac{3q}{2} - \frac{3}{2}) - q\nu - \nu + \rho \quad (6q \leq q\nu + \nu) \\
&= \nu(\frac{q}{2} - \frac{5}{2}) + \rho \geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture in this case.

Thus, Wilf's Conjecture holds if $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$. \square

Next we will generalize a result for Sammartano ([9]) and show that Wilf's Conjecture holds for numerical semigroups satisfying $(2 + \frac{1}{q})\nu \geq m$, using Lemma 3.7, Corollary 3.9, Theorem 3.11, Lemma 3.13 and Corollary 3.5.

Theorem 3.14 Let S be a numerical semigroup with multiplicity m , embedding dimension ν and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $(2 + \frac{1}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof.

- If $m - \nu \leq 3$, then by Lemma 3.7 Wilf's Conjecture holds.
- If $m - \nu = 4$, then by Corollary 3.9 Wilf's Conjecture holds.
- If $m - \nu = 5$, then by Theorem 3.11 Wilf's Conjecture holds.
- If $m - \nu = 6$ and $(2 + \frac{1}{q})\nu \geq m$, then by Lemma 3.13 Wilf's Conjecture holds.
- If $m - \nu > 6$ and $(2 + \frac{1}{q})\nu \geq m$, then by Corollary 3.5 Wilf's Conjecture holds. \square

Example 3.15 Consider the following numerical semigroup $S = \langle 13, 15, 17, 19, 21, 27 \rangle$. Note that $2\nu < m$. We have $(2 + \frac{1}{q})\nu = (2 + \frac{1}{4})6 \geq 13 = m$. Thus the conditions of Theorem 3.14 are valid, so S satisfies Wilf's Conjecture.

Corollary 3.16 Let S be a numerical semigroup with multiplicity m and conductor $f + 1 = qm - \rho$ for some $q \in \mathbb{N}$, $0 \leq \rho \leq m - 1$. If $m \leq 8 + \frac{4}{q}$, then S satisfies Wilf's Conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's Conjecture ([3]). Hence, we can suppose that $\nu \geq 4$. Thus, $(2 + \frac{1}{q})\nu \geq (2 + \frac{1}{q})4 \geq m$. By using Theorem 3.14 S satisfies Wilf's Conjecture. \square

4 Numerical semigroups with $w_{m-1} \geq w_{\alpha-1} + w_{\alpha}$ and $(\frac{\alpha+3}{3})\nu \geq m$

In this section, we will show that if S is a numerical Semigroup such that $w_{m-1} \geq w_{\alpha-1} + w_{\alpha}$ and $(\frac{\alpha+3}{3})\nu \geq m$, then S satisfies Wilf's Conjecture.

Theorem 4.1 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_{\alpha}$ for some $1 < \alpha < m - 1$. If $(\frac{\alpha+3}{3})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. We may assume that $\rho \geq \frac{(3-q)\alpha m}{2\alpha+6}$. Indeed, if $0 \leq \rho < \frac{(3-q)\alpha m}{2\alpha+6}$, then $q < 3$ and Wilf's conjecture holds for this case ([6]). We are going to show that S satisfies Wilf's Conjecture by means of Proposition 2.3. We have,

$$\begin{aligned}
\sum_{j=1}^{\alpha} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) &= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=1}^{\alpha} \lfloor \frac{w_{j-1} + \rho}{m} \rfloor(j\nu - m) \\
&= \sum_{j=1}^{\alpha} \lfloor \frac{w_j + \rho}{m} \rfloor(j\nu - m) - \sum_{j=0}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor((j+1)\nu - m) \\
&= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_0 + \rho}{m} \rfloor(\nu - m) - \sum_{j=1}^{\alpha-1} \lfloor \frac{w_j + \rho}{m} \rfloor\nu \\
&= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor\nu - \sum_{j=1}^{\alpha-2} \lfloor \frac{w_j + \rho}{m} \rfloor\nu \\
&\geq \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor\nu - \sum_{j=1}^{\alpha-2} \left(\frac{\lfloor \frac{w_{\alpha} + \rho}{m} \rfloor + \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor}{2} \right)\nu \\
&= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor(\alpha\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor\nu - (\lfloor \frac{w_{\alpha} + \rho}{m} \rfloor + \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor) \frac{(\alpha-2)\nu}{2} \\
&= \lfloor \frac{w_{\alpha} + \rho}{m} \rfloor((\frac{\alpha+2}{2})\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor(\frac{\alpha\nu}{2}).
\end{aligned}$$

By (3.2), we have $\sum_{j=\alpha+1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) \geq (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha + 1)\nu - m)$.

Since $w_{m-1} \geq w_{\alpha-1} + w_\alpha$, by Lemma 3.1, it follows that $\lfloor \frac{w_{m-1} + \rho}{m} \rfloor \geq \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor - 1$. Let $x = \lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor$. Then, $\lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor = q - x$ and $x \geq -1$. Now using $\rho \geq \frac{(3-q)\alpha m}{2\alpha+6}$ and $(\frac{\alpha+3}{3})\nu \geq m$, we get

$$\begin{aligned}
\sum_{j=1}^{m-1} (\lfloor \frac{w_j + \rho}{m} \rfloor - \lfloor \frac{w_{j-1} + \rho}{m} \rfloor)(j\nu - m) + \rho &\geq \lfloor \frac{w_\alpha + \rho}{m} \rfloor ((\frac{\alpha+2}{2})\nu - m) - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor (\frac{\alpha\nu}{2}) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha + 1)\nu - m) + \rho \\
&= \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor (\frac{-\alpha\nu}{2} + (\alpha + 1)\nu - m - ((\alpha + 1)\nu - m)) \\
&\quad + \lfloor \frac{w_\alpha + \rho}{m} \rfloor ((\frac{\alpha+2}{2})\nu - m) + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha + 1)\nu - m) \\
&\quad + \rho \\
&= (\lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor + \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\frac{\alpha+2}{2})\nu - m) \\
&\quad + (\lfloor \frac{w_{m-1} + \rho}{m} \rfloor - \lfloor \frac{w_{\alpha-1} + \rho}{m} \rfloor - \lfloor \frac{w_\alpha + \rho}{m} \rfloor)((\alpha + 1)\nu - m) + \rho \\
&= (q - x)((\frac{\alpha+2}{2})\nu - m) + x((\alpha + 1)\nu - m) + \rho \\
&= \nu(q + \frac{q\alpha}{2} + \frac{x\alpha}{2}) - qm + \rho \\
&\geq \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - qm + \frac{(3-q)\alpha m}{2\alpha+6} \\
&= \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - m(\frac{q(2\alpha+6)+(q-3)\alpha}{2\alpha+6}) \\
&= \nu(q + \frac{q\alpha}{2} - \frac{\alpha}{2}) - m(\frac{3q}{\alpha+3} + \frac{3q\alpha}{2(\alpha+3)} - \frac{3\alpha}{2(\alpha+3)}) \\
&= (q + \frac{q\alpha}{2} - \frac{\alpha}{2})(\frac{3}{\alpha+3})((\frac{\alpha+3}{3})\nu - m) \\
&\geq 0.
\end{aligned}$$

Using Proposition 2.3, we get that S satisfies Wilf's Conjecture. \square

Example 4.2 Consider the following numerical semigroup $S = \langle 22, 23, 25, 27, 29, 31, 33 \rangle$. Note that $3\nu < m$. We have $w_6 = 33$, $w_7 = 46$ and $w_{m-1} = 87$ and i.e. $w_{m-1} \geq w_6 + w_7$. Moreover, $(\frac{\alpha+3}{3})\nu = (\frac{7+3}{3})7 \geq 22 = m$, thus the conditions of Theorem 4.1 are valid. Hence, S satisfies Wilf's Conjecture. \square

The following Corollary 4.3 is an extension for Corollary 3.5 using Theorems 3.2 and 4.1.

Corollary 4.3 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Suppose that $m - \nu \geq \frac{\alpha(\alpha-1)}{2} - 1$ for some $7 \leq \alpha \leq m - 2$. If $(2 + \frac{\alpha-3}{q})\nu \geq m$, then S satisfies Wilf's Conjecture.

Proof. Since $m - \nu \geq \frac{\alpha(\alpha-1)}{2} - 1$, then by Lemma 3.10 we have $w_{m-1} \geq w_1 + w_\alpha$ or $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. Suppose that $w_{m-1} \geq w_1 + w_\alpha$. Since $(2 + \frac{\alpha-3}{q})\nu \geq m$, by applying Theorem 3.2, S satisfies Wilf's Conjecture. Now suppose that $w_{m-1} \geq w_{\alpha-2} + w_{\alpha-1}$. We may assume that $q \geq 4$ ($q \leq 3$ is solved [6], [4]). Then for $\alpha \geq 7$

we have, $(\frac{\alpha-1+3}{3})\nu \geq (2 + \frac{\alpha-3}{q})\nu$. Consequently, $(\frac{\alpha-1+3}{3})\nu \geq m$. Now by applying Theorem 4.1, S satisfies Wilf's Conjecture. \square

As a direct consequence of Theorem 4.1, we get the following Corollary.

Corollary 4.4 Let S be a numerical semigroup with multiplicity m and embedding dimension ν . Let $w_0 = 0 < w_1 < w_2 < \dots < w_{m-1}$ be the elements of $\text{Ap}(S, m)$. Suppose that $w_{m-1} \geq w_{\alpha-1} + w_\alpha$ for some $1 < \alpha < m-1$. If $m \leq \frac{4(\alpha+3)}{3}$, then S satisfies Wilf's Conjecture.

Proof. If $\nu < 4$, then S satisfies Wilf's Conjecture ([3]). Hence, we can suppose that $\nu \geq 4$. Thus, $(\frac{\alpha+3}{3})(\nu) \geq \frac{4(\alpha+3)}{3} \geq m$. By applying Theorem 4.1 S satisfies Wilf's Conjecture. \square

References

- [1] V. Barucci, On propinquity of numerical semigroups and one-dimensional local Cohen Macaulay rings, Journal of Commutative algebra and its applications, Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications (2009), pp. 49–60.
- [2] M. Bras-Amorós, Fibonacci-like behavior of the number of numerical semigroups of a given genus, Semigroup Forum, vol 76, n°2 (2008), pp. 379–384.
- [3] D. Dobbs, G. Matthews, On a question of Wilf concerning numerical semigroups, Focus on commutative rings research (2006), pp. 193–202.
- [4] S. Eliahou, Wilf's Conjecture and Macaulay's theorem, preprint.
<http://www.ugr.es/imns2010/2016/preprints/eliahou-imns2016.pdf>
- [5] R. Fröberg, C. Gottlieb, R. Häggkvist, On numerical semigroups, Semigroup forum, vol 35 (1986-1987) pp. 63–83.
- [6] N. Kaplan, Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure and Applied Algebra, vol 216, n°5 (2012), pp. 1016–1032.
- [7] J. L. Ramírez-Alfonsín, Complexity of the Frobenius problem, Combinatorica, vol 16, n°1 (1996), pp. 143–147.
- [8] P.A. García-Sánchez, J. C. Rosales, Numerical semigroups, Developments in Mathematics, 20. Springer, New York, (2009).
- [9] A. Sammartano, Numerical semigroups with large embedding dimension satisfy Wilf's conjecture, Semigroup Forum, vol 85 n°3 (2012), pp. 439–447.
- [10] J. J. Sylvester, Mathematical questions with their solutions, Educational Times, vol 41, (1884), pp. 21.
- [11] H. S. Wilf, A circle-of-lights algorithm for the "money-changing problem", The Amer. Math. Monthly, vol 85, n°7 (1978) pp. 562–565.
- [12] A. Zhai, An asymptotic result concerning a question of Wilf, arXiv :1111.2779 (2011).